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Mathematical structure of Rabi oscillations in the strong coupling regime

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Abstract

In this paper, we generalize the Jaynes–Cummings Hamiltonian by making use of some operators based on Lie algebras $su(1,1)$ and $su(2)$, and study a mathematical structure of Rabi floppings of these models in the strong coupling regime. We show that Rabi frequencies are given by matrix elements of generalized coherent operators (Fujii K 2002 *Preprint* quant-ph/0202081) under the rotating-wave approximation. In the first half, we make a general review of coherent operators and generalized coherent ones based on Lie algebras $su(1,1)$ and $su(2)$. In the latter half, we carry out a detailed examination of Frasca (Frasca M 2001 *Preprint* quant-ph/0111134) and generalize his method, and moreover present some related problems. We also apply our results to the construction of controlled unitary gates in quantum computation. Lastly, we make a brief comment on application to holonomic quantum computation.

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1. Introduction

Coherent states or generalized coherent states play an important role in quantum physics, in particular, quantum optics, see [1, 2]. They also play an important role in mathematical physics, see [3]. For example, they are very useful in performing stationary phase approximations to path integrals [4–6].

Coherent operators which produce coherent states are very useful because they are unitary and easy to handle. The basic reason is probably that they are subject to the elementary Baker–Campbell–Hausdorff (BCH) formula. Many basic properties of them are well known, see [3] or [8].

Generalized coherent operators which produce generalized coherent states are also useful. But they are not so easy to handle in spite of having the disentangling one corresponding to the elementary BCH formula. In [7, 14] the author determined all matrix elements of generalized coherent operators based on Lie algebras $su(1,1)$ and $su(2)$. They are interesting in themselves, but moreover have a very interesting application.

In [12] Frasca dealt with the Jaynes–Cummings model which describes a two-level atom interacting with a single radiation mode (see [10] for a general review) in the strong coupling regime (not weak coupling one!) and showed that Rabi frequencies are obtained by matrix elements of the coherent operator under the rotating-wave approximation. His aim was to explain the recent experimental finding on Josephson junctions [11].

This is an interesting result and moreover his method can be widely generalized. See also [13] for another example dealt with in the strong coupling regime.

In this paper, we generalize the Jaynes–Cummings Hamiltonian by making use of some operators based on Lie algebras $su(1,1)$ and $su(2)$, and study a mathematical structure of Rabi floppings of these extended models in the strong coupling regime.

We show that (generalized) Rabi frequencies are also given by matrix elements of generalized coherent operators under the rotating-wave approximation. We believe that the results will give a new aspect to quantum optics or mathematical physics.

We also apply our results to the construction of controlled unitary gates in quantum computation in the last section.

Finally we discuss an application to holonomic quantum computation, but our discussion is not complete.

2. Coherent and generalized coherent operators

2.1. Coherent operator

Let a ($a^\dagger$) be the annihilation (creation) operator of the harmonic oscillator. If we set $N \equiv a^\dagger a$ (number operator), then

$$[N, a^\dagger] = a^\dagger \qquad [N, a] = -a \qquad [a^\dagger, a] = -\mathbf{1}. \tag{1}$$

Let $\mathcal{H}$ be a Fock space generated by a and $a^\dagger$, and $\{|n\rangle | n \in \mathbf{N} \cup \{0\}\}$ be its basis. The actions of a and $a^\dagger$ on $\mathcal{H}$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad N|n\rangle = n|n\rangle \tag{2}$$

where $|0\rangle$ is a normalized vacuum ($a|0\rangle = 0$ and $\langle 0|0\rangle = 1$). From (2) states $|n\rangle$ for $n \geqslant 1$ are given by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{3}$$

These states satisfy the orthogonality and completeness conditions

$$\langle m|n\rangle = \delta_{mn} \qquad \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}. \tag{4}$$

Definition. *We call a state*

$$|z\rangle = \mathrm{e}^{za^\dagger - \bar{z}a}|0\rangle \equiv U(z)|0\rangle \qquad \textit{for} \quad z \in \mathbf{C} \tag{5}$$

the coherent state.

2.2. Generalized coherent operator based on su(1, 1)

Let us state generalized coherent operators and states based on $su(1,1)$.

We consider a spin $K(>0)$ representation of $su(1,1) \subset sl(2,\mathbf{C})$ and set its generators $\{K_+, K_-, K_3\}((K_+)^\dagger = K_-)$,

$$[K_3, K_+] = K_+ \qquad [K_3, K_-] = -K_- \qquad [K_+, K_-] = -2K_3. \tag{6}$$

We note that this (unitary) representation is necessarily infinite dimensional. The Fock space on which $\{K_+, K_-, K_3\}$ act is $\mathcal{H}_K \equiv \{|K,n\rangle | n \in \mathbf{N} \cup \{0\}\}$ and whose actions are

$$K_+|K,n\rangle = \sqrt{(n+1)(2K+n)}|K,n+1\rangle \qquad K_-|K,n\rangle = \sqrt{n(2K+n-1)}|K,n-1\rangle$$
$$K_3|K,n\rangle = (K+n)|K,n\rangle \tag{7}$$

where $|K,0\rangle$ is a normalized vacuum ($K_-|K,0\rangle = 0$ and $\langle K,0|K,0\rangle = 1$). We have written $|K,0\rangle$ instead of $|0\rangle$ to emphasize the spin K representation, see [4]. From (7), states $|K,n\rangle$ are given by

$$|K,n\rangle = \frac{(K_+)^n}{\sqrt{n!(2K)_n}}|K,0\rangle \tag{8}$$

where $(a)_n$ is Pochammer's notation $(a)_n \equiv a(a+1)\cdots(a+n-1)$. These states satisfy the orthogonality and completeness conditions

$$\langle K,m|K,n\rangle = \delta_{mn} \qquad \sum_{n=0}^{\infty} |K,n\rangle\langle K,n| = \mathbf{1}_K. \tag{9}$$

Now let us consider a generalized version of coherent states:

Definition. *We call a state*

$$|z\rangle = V(z)|K,0\rangle \equiv \mathrm{e}^{zK_+ - \bar{z}K_-}|K,0\rangle \qquad \textit{for} \quad z \in \mathbf{C} \tag{10}$$

the generalized coherent state (or the coherent state of Perelomov's type based on $su(1,1)$ *in our terminology).*

Here let us construct an example of this representation. First we set

$$K_+ \equiv \tfrac{1}{2}(a^\dagger)^2 \qquad K_- \equiv \tfrac{1}{2}a^2 \qquad K_3 \equiv \tfrac{1}{2}\left(a^\dagger a + \tfrac{1}{2}\right) \tag{11}$$

then it is easy to check that these satisfy the commutation relations (6). That is, the set $\{K_+, K_-, K_3\}$ gives a unitary representation of $su(1,1)$ with spin $K = 1/4$ and $3/4$ [3]. Now we also call an operator

$$S(z) = \mathrm{e}^{\frac{1}{2}\{z(a^\dagger)^2 - \bar{z}a^2\}} \qquad \text{for} \quad z \in \mathbf{C} \tag{12}$$

the squeezed operator, see [3].

2.3. Generalized coherent operator based on su(2)

Let us state generalized coherent operators and states based on $su(2)$.

We consider a spin $J\ (>0)$ representation of $su(2) \subset sl(2,\mathbf{C})$ and set its generators $\{J_+, J_-, J_3\}\ ((J_+)^\dagger = J_-)$,

$$[J_3, J_+] = J_+ \qquad [J_3, J_-] = -J_- \qquad [J_+, J_-] = 2J_3. \tag{13}$$

We note that this (unitary) representation is necessarily finite dimensional. The Fock space on which $\{J_+, J_-, J_3\}$ act is $\mathcal{H}_J \equiv \{|J,n\rangle | 0 \leqslant n \leqslant 2J\}$ and its actions are

$$J_+|J,n\rangle = \sqrt{(n+1)(2J-n)}|J,n+1\rangle \qquad J_-|J,n\rangle = \sqrt{n(2J-n+1)}|J,n-1\rangle$$
$$J_3|J,n\rangle = (-J+n)|J,n\rangle \tag{14}$$

where $|J, 0\rangle$ is a normalized vacuum ($J_-|J, 0\rangle = 0$ and $\langle J, 0|J, 0\rangle = 1$). We have written $|J, 0\rangle$ instead of $|0\rangle$ to emphasize the spin J representation, see [4]. From (14), states $|J, n\rangle$ are given by

$$|J, n\rangle = \frac{(J_+)^n}{\sqrt{n!_{2J}P_n}}|J, 0\rangle. \tag{15}$$

These states satisfy the orthogonality and completeness conditions

$$\langle J, m|J, n\rangle = \delta_{mn} \qquad \sum_{n=0}^{2J} |J, n\rangle\langle J, n| = \mathbf{1}_J. \tag{16}$$

Now let us consider a generalized version of coherent states:

Definition. *We call a state*

$$|z\rangle = W(z)|J, 0\rangle \equiv \mathrm{e}^{zJ_+ - \bar{z}J_-}|J, 0\rangle \qquad \textit{for} \quad z \in \mathbf{C}. \tag{17}$$

the generalized coherent state (or the coherent state of Perelomov's type based on $su(2)$ *in our terminology).*

A comment is in order. We can construct the spin K and J representations by making use of Schwinger's boson method. But we do not repeat them here, see for example [7].

3. Matrix elements of coherent and generalized coherent operators . . . [14]

3.1. Matrix elements of coherent operators

We list matrix elements of coherent operators $U(z)$.

The matrix elements. The matrix elements of $U(z)$ are:

$$\text{(i)} \quad n \leqslant m \quad \langle n|U(z)|m\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\sqrt{\frac{n!}{m!}}(-\bar{z})^{m-n}L_n{}^{(m-n)}(|z|^2) \tag{18}$$

$$\text{(ii)} \quad n \geqslant m \quad \langle n|U(z)|m\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\sqrt{\frac{m!}{n!}}z^{n-m}L_m{}^{(n-m)}(|z|^2) \tag{19}$$

where $L_n{}^{(\alpha)}$ is the associated Laguerre's polynomial defined by

$$L_k{}^{(\alpha)}(x) = \sum_{j=0}^{k}(-1)^j\binom{k+\alpha}{k-j}\frac{x^j}{j!}. \tag{20}$$

In particular $L_k \equiv L_k{}^{(0)}$ is the usual Laguerre's polynomial and these are related to diagonal elements of $U(z)$.

3.2. Matrix elements of coherent operators based on su(1, 1)

We list matrix elements of $V(z)$ coherent operators based on $su(1, 1)$. In this case it is always $2K > 1$ ($2K = 1$ under some regularization).

The matrix elements. The matrix elements of $V(z)$ are

$$\begin{aligned}\text{(i)} \quad n \leqslant m \quad \langle K, n|V(z)|K, m\rangle &= \sqrt{\frac{n!m!}{(2K)_n(2K)_m}}(-\bar{\kappa})^{m-n}(1+|\kappa|^2)^{-K-\frac{n+m}{2}} \\ &\times \sum_{j=0}^{n}(-1)^{n-j}\frac{\Gamma(2K+m+n-j)}{\Gamma(2K)(m-j)!(n-j)!j!}(1+|\kappa|^2)^j(|\kappa|^2)^{n-j}\end{aligned} \tag{21}$$

(ii) $n \geqslant m$ $\quad \langle K, n|V(z)|K, m\rangle = \sqrt{\frac{n!m!}{(2K)_n(2K)_m}}\kappa^{n-m}(1+|\kappa|^2)^{-K-\frac{n+m}{2}}$

$$\times \sum_{j=0}^{m}(-1)^{m-j}\frac{\Gamma(2K+m+n-j)}{\Gamma(2K)(m-j)!(n-j)!j!}(1+|\kappa|^2)^j(|\kappa|^2)^{m-j} \tag{22}$$

where

$$\kappa \equiv \frac{\sinh(|z|)}{|z|}z = \cosh(|z|)\zeta. \tag{23}$$

The author does not know whether or not the right-hand sides of (21) and (22) could be written by making use of some special functions such as generalized Laguerre's functions in (20). Therefore, we set temporarily

$$F_m{}^{(n-m)}(x:2K) = \sum_{j=0}^{m}(-1)^{m-j}\frac{\Gamma(2K+m+n-j)}{\Gamma(2K)(m-j)!(n-j)!j!}(1+x)^j x^{m-j} \tag{24}$$

and $F_m{}^{(0)}(x; 2K) = F_m(x; 2K)$.

3.3. Matrix elements of coherent operators based on su(2)

We list matrix elements of $W(z)$ coherent operators based on $su(2)$. In this case it is always $2J \in \mathbf{N}$.

Matrix elements. The matrix elements of $W(z)$ are

(i) $n \leqslant m$ $\quad \langle J, n|W(z)|J, m\rangle = \sqrt{\frac{n!m!}{{}_{2J}P_{n\,2J}P_m}}(-\bar{\kappa})^{m-n}(1-|\kappa|^2)^{J-\frac{n+m}{2}}$

$$\times \sum_{j=0}^{n}{}_{*}(-1)^{n-j}\frac{(2J)!}{(2J-m-n+j)!(m-j)!(n-j)!j!}(1-|\kappa|^2)^j(|\kappa|^2)^{n-j} \tag{25}$$

(ii) $n \geqslant m$ $\quad \langle J, n|W(z)|J, m\rangle = \sqrt{\frac{n!m!}{{}_{2J}P_{n\,2J}P_m}}\kappa^{n-m}(1-|\kappa|^2)^{J-\frac{n+m}{2}}$

$$\times \sum_{j=0}^{m}{}_{*}(-1)^{m-j}\frac{(2J)!}{(2J-m-n+j)!(m-j)!(n-j)!j!}(1-|\kappa|^2)^j(|\kappa|^2)^{m-j} \tag{26}$$

where

$$\kappa \equiv \frac{\sin(|z|)}{|z|}z = \cos(|z|)\eta. \tag{27}$$

Here $\sum_{*}$ means a summation over j satisfying $2J - m - n + j \geqslant 0$.

The author does not know whether or not the right-hand sides of (25) and (26) could be written by making use of some special functions. We temporarily set

$$F_m{}^{(n-m)}(x:2J) = \sum_{j=0}^{m}{}_{*}(-1)^{m-j}\frac{(2J)!}{(2J-m-n+j)!(m-j)!(n-j)!j!}(1-x)^j x^{m-j} \tag{28}$$

and $F_m{}^{(0)}(x; 2J) = F_m(x; 2J)$.

4. Jaynes–Cummings model in the strong coupling regime

In [12] Frasca treated the Jaynes–Cummings model and developed some method to calculate Rabi frequencies in the strong coupling regime. We in this section generalize the model and method, and show that Rabi frequencies in our extended model are given by matrix elements of generalized coherent operators under the rotating-wave approximation. This gives a unified approach to them.

Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be Pauli matrices and $\mathbf{1}_2$ a unit matrix:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{29}$$

The Hamiltonian adopted in [12] is

$$\text{(N)} \quad H_N = \omega \mathbf{1}_2 \otimes a^\dagger a + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1} + g\sigma_1 \otimes (a^\dagger + a) \tag{30}$$

where ω is the frequency of the radiation mode, Δ the separation between the two levels of the atom, g the coupling between the radiation field and the atom.

Moreover we want to treat the following Hamiltonians (our extension):

$$\text{(K)} \quad H_K = \omega \mathbf{1}_2 \otimes K_3 + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1}_K + g\sigma_1 \otimes (K_+ + K_-) \tag{31}$$

$$\text{(J)} \quad H_J = \omega \mathbf{1}_2 \otimes J_3 + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1}_J + g\sigma_1 \otimes (J_+ + J_-). \tag{32}$$

To treat these three cases at the same time we set

$$\{L_+, L_-, L_3\} = \begin{cases} \text{(N)} & \{a^\dagger, a, N\} \\ \text{(K)} & \{K_+, K_-, K_3\} \\ \text{(J)} & \{J_+, J_-, J_3\} \end{cases} \tag{33}$$

and

$$H = H_0 + V = \omega \mathbf{1}_2 \otimes L_3 + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1}_L + g\sigma_1 \otimes (L_+ + L_-) \tag{34}$$

where we have written H instead of H_L for simplicity.

Mysteriously enough we cannot solve these simple models completely (maybe non-integrable), nevertheless we have found that these models have a very rich structure.

For these (non-integrable) models we usually have two perturbation approaches:

Weak coupling regime $(0 < g \ll \Delta)$

$$H_0 = \omega \mathbf{1}_2 \otimes L_3 + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1}_L \qquad V = g\sigma_1 \otimes (L_+ + L_-). \tag{35}$$

Strong coupling regime $(0 < \Delta \ll g)$

$$H_0 = \omega \mathbf{1}_2 \otimes L_3 + g\sigma_1 \otimes (L_+ + L_-) \qquad V = \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1}_L. \tag{36}$$

In the following we consider only the strong coupling regime (see [10] for the weak one). First let us solve H_0 which is a relatively easy task.

Let W be a Walsh–Hadamard matrix

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = W^{-1}$$

then we can diagonalize σ_1 by using this H as $\sigma_1 = W\sigma_3 W^{-1}$. The eigenvalues of σ_1 are $\{1, -1\}$ with eigenvectors

$$|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad |-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \quad \Longrightarrow \quad |\lambda\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\\lambda\end{pmatrix}.$$

We note that

$$\begin{aligned}
|1\rangle\langle 1| &= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} = W\begin{pmatrix}1 & \\ & 0\end{pmatrix}W^{-1}\\
|-1\rangle\langle -1| &= \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} = W\begin{pmatrix}0 & \\ & 1\end{pmatrix}W^{-1}\\
\Longrightarrow \quad |\lambda\rangle\langle\lambda| &= \frac{1}{2}\begin{pmatrix}1 & \lambda\\\lambda & 1\end{pmatrix} = W\begin{pmatrix}\frac{1+\lambda}{2} & \\ & \frac{1-\lambda}{2}\end{pmatrix}W^{-1}.
\end{aligned}$$

Then we have

$$\begin{aligned}
H_0 &= (W\otimes \mathbf{1}_L)(\omega\mathbf{1}_2\otimes L_3 + g\sigma_3\otimes(L_+ + L_-))(W^{-1}\otimes\mathbf{1}_L)\\
&= (W\otimes\mathbf{1}_L)\begin{pmatrix}\omega L_3 + g(L_+ + L_-) & \\ & \omega L_3 - g(L_+ + L_-)\end{pmatrix}(W^{-1}\otimes\mathbf{1}_L).\\
&= |1\rangle\langle 1|\otimes\{\omega L_3 + g(L_+ + L_-)\} + |-1\rangle\langle -1|\otimes\{\omega L_3 - g(L_+ + L_-)\}\\
&= \sum_\lambda |\lambda\rangle\langle\lambda|\otimes\{\omega L_3 + \lambda g(L_+ + L_-)\}\\
&= \sum_\lambda |\lambda\rangle\langle\lambda|\otimes\left\{\mathrm{e}^{-\frac{\lambda x}{2}(L_+ - L_-)}(\Omega L_3)\,\mathrm{e}^{\frac{\lambda x}{2}(L_+ - L_-)}\right\}\\
&= \sum_\lambda \left(|\lambda\rangle\otimes\mathrm{e}^{-\frac{\lambda x}{2}(L_+ - L_-)}\right)(\Omega L_3)\left(\langle\lambda|\otimes\mathrm{e}^{\frac{\lambda x}{2}(L_+ - L_-)}\right)
\end{aligned} \tag{37}$$

where we have used the following:

Key formulae. For $\lambda = \pm 1$ we have

$$\text{(N)}\quad \omega a^\dagger a + \lambda g(a^\dagger + a) = \Omega\,\mathrm{e}^{-\frac{\lambda x}{2}(a^\dagger - a)}\left(N - \frac{g^2}{\omega^2}\right)\mathrm{e}^{\frac{\lambda x}{2}(a^\dagger - a)}$$
$$\text{where}\quad \Omega = \omega \quad x = 2g/\omega \tag{38}$$

$$\text{(K)}\quad \omega K_3 + \lambda g(K_+ + K_-) = \Omega\,\mathrm{e}^{-\frac{\lambda x}{2}(K_+ - K_-)}K_3\,\mathrm{e}^{\frac{\lambda x}{2}(K_+ - K_-)}$$
$$\text{where}\quad \Omega = \omega\sqrt{1-(2g/\omega)^2}\quad x = \tanh^{-1}(2g/\omega) \tag{39}$$

$$\text{(J)}\quad \omega J_3 + \lambda g(J_+ + J_-) = \Omega\,\mathrm{e}^{-\frac{\lambda x}{2}(J_+ - J_-)}J_3\,\mathrm{e}^{\frac{\lambda x}{2}(J_+ - J_-)}$$
$$\text{where}\quad \Omega = \omega\sqrt{1+(2g/\omega)^2}\quad x = \tan^{-1}(2g/\omega). \tag{40}$$

The proof is not difficult, so we leave it to the readers. That is, we could diagonalize the Hamiltonian H_0. This is two-fold degenerate and its eigenvalues and eigenvectors are given, respectively, as

$$(\text{Eigenvalues, Eigenvectors}) = \begin{cases}\text{(N)} & \omega n - \frac{g^2}{\omega} & |\lambda\rangle\otimes\mathrm{e}^{-\frac{\lambda x}{2}(a^\dagger - a)}|n\rangle\\ \text{(K)} & \Omega(K+n) & |\lambda\rangle\otimes\mathrm{e}^{-\frac{\lambda x}{2}(K_+ - K_-)}|K,n\rangle\\ \text{(J)} & \Omega(-J+n) & |\lambda\rangle\otimes\mathrm{e}^{-\frac{\lambda x}{2}(J_+ - J_-)}|J,n\rangle\end{cases} \tag{41}$$

for $\lambda = \pm 1$ and $n \in \mathbf{N}\cup\{0\}$. For later convenience we set

$$\text{Eigenvalues} = \{E_n\} \qquad \text{Eigenvectors} = \{|\{\lambda, n\}\rangle\}. \tag{42}$$

Then (37) can be written as

$$H_0 = \sum_{\lambda} \sum_{n} E_n |\{\lambda, n\}\rangle\langle\{\lambda, n\}|. \tag{43}$$

Next we would like to solve the following Schrödinger equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi = H\Psi = \left(H_0 + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1}_L\right)\Psi \tag{44}$$

where we have set $\hbar = 1$ for simplicity. To solve this equation we apply the method of constant variation. First let us solve

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi = H_0\Psi \tag{45}$$

whose general solution is given by

$$\Psi(t) = U_0(t)\Psi_0 = \mathrm{e}^{-\mathrm{i}tH_0}\Psi_0 \tag{46}$$

where Ψ_0 is a constant state. It is easy to see from (43)

$$U_0(t) = \mathrm{e}^{-\mathrm{i}tH_0} = \sum_{\lambda} \sum_{n} \mathrm{e}^{-\mathrm{i}tE_n} |\{\lambda, n\}\rangle\langle\{\lambda, n\}|. \tag{47}$$

The method of constant variation goes as follows. Changing $\Psi_0 \longrightarrow \Psi_0(t)$, we insert (46) into (A.1). After some algebra we obtain

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_0 = \frac{\Delta}{2}{U_0}^{\dagger}(\sigma_3 \otimes \mathbf{1}_L)U_0\Psi_0. \tag{48}$$

We have only to solve this equation. If we set

$$H_F = \frac{\Delta}{2}{U_0}^{\dagger}(\sigma_3 \otimes \mathbf{1}_L)U_0 \tag{49}$$

then we have easily from (47)

$$\begin{aligned} H_F &= \frac{\Delta}{2}\sum_{\lambda,\mu}\sum_{m,n} \mathrm{e}^{\mathrm{i}t(E_m-E_n)}\langle\{\lambda, m\}|(\sigma_3 \otimes \mathbf{1}_L)|\{\mu, n\}\rangle|\{\lambda, m\}\rangle\langle\{\mu, n\}| \\ &= \frac{\Delta}{2}\sum_{\lambda}\sum_{m,n} \mathrm{e}^{\mathrm{i}t\Omega(m-n)}\langle\langle m|\,\mathrm{e}^{\lambda x(L_+-L_-)}|n\rangle\rangle|\{\lambda, m\}\rangle\langle\{-\lambda, n\}| \end{aligned} \tag{50}$$

where we have used the relation $\langle\lambda|\sigma_3 = \langle-\lambda|$. Recall that $|n\rangle\rangle$ is respectively

$$|n\rangle\rangle = \begin{cases} (\mathrm{N}) & |n\rangle \\ (\mathrm{K}) & |K, n\rangle \\ (\mathrm{J}) & |J, n\rangle \end{cases}$$

At this stage we meet *matrix elements of the coherent and generalized coherent operators* $\mathrm{e}^{\lambda x(L_+-L_-)}$ in section 3 ($z = \bar{z} = \lambda x$).

Here we divide H_F into two parts

$$H_F = H_F' + H_F''$$

where

$$H_F' = \frac{\Delta}{2}\sum_{\lambda}\sum_{n}\langle\langle n|\,\mathrm{e}^{\lambda x(L_+-L_-)}|n\rangle\rangle|\{\lambda, n\}\rangle\langle\{-\lambda, n\}| \tag{51}$$

$$H_F'' = \frac{\Delta}{2}\sum_{\lambda}\sum_{\substack{m,n \\ m\neq n}} \mathrm{e}^{\mathrm{i}t\Omega(m-n)}\langle\langle m|\,\mathrm{e}^{\lambda x(L_+-L_-)}|n\rangle\rangle|\{\lambda, m\}\rangle\langle\{-\lambda, n\}|. \tag{52}$$

Noting

$$\langle\langle n|\,\mathrm{e}^{x(L_+-L_-)}|n\rangle\rangle = \langle\langle n|\,\mathrm{e}^{-x(L_+-L_-)}|n\rangle\rangle$$

by the results in section 3, $H_F{}'$ can be written as

$$H_F' = \frac{\Delta}{2}\sum_n \langle\langle n|\,\mathrm{e}^{x(L_+-L_-)}|n\rangle\rangle\,\{|\{1,n\}\rangle\langle\{-1,n\}| + |\{-1,n\}\rangle\langle\{1,n\}|\}$$

from which we can diagonalize H_F' as

$$H_F' = \frac{\Delta}{2}\sum_n\sum_\sigma \langle\langle n|\,\mathrm{e}^{x(L_+-L_-)}|n\rangle\rangle\sigma|\{\sigma,\psi_n\}\rangle\langle\{\sigma,\psi_n\}| \tag{53}$$

if we define a new basis

$$|\{\sigma,\psi_n\}\rangle = \frac{1}{\sqrt{2}}(\sigma|\{1,n\}\rangle + |\{-1,n\}\rangle) \qquad \sigma = \pm 1.$$

These states can be seen as so-called Schrödinger cat states [15]. From these we have

$$|\{1,n\}\rangle = \frac{1}{\sqrt{2}}\{|\{1,\psi_n\}\rangle - |\{-1,\psi_n\}\rangle\}$$
$$|\{-1,n\}\rangle = \frac{1}{\sqrt{2}}\{|\{1,\psi_n\}\rangle + |\{-1,\psi_n\}\rangle\}.$$

Inserting these equations into (52) and taking some algebras we obtain

$$H_F'' = \frac{\Delta}{2}\sum_{\substack{m,n\\ m\neq n}}\sum_{\sigma,\sigma'} \mathrm{e}^{\mathrm{i}t\Omega(m-n)}\Bigg\{\langle\langle m|\,\mathrm{e}^{x(L_+-L_-)}|n\rangle\rangle\frac{\sigma}{2}|\{\sigma,\psi_m\}\rangle\langle\{\sigma',\psi_n\}|$$
$$+\,\langle\langle m|\,\mathrm{e}^{-x(L_+-L_-)}|n\rangle\rangle\frac{\sigma'}{2}|\{\sigma,\psi_m\}\rangle\langle\{\sigma',\psi_n\}|\Bigg\}. \tag{54}$$

For simplicity in (53) we set in the following

$$E_{n,\sigma} = \frac{\Delta}{2}\sigma\langle\langle n|\,\mathrm{e}^{x(L_+-L_-)}|n\rangle\rangle \tag{55}$$

then

$$E_{n,\sigma} = \begin{cases} (\mathrm{N}) & \frac{\Delta}{2}\sigma\,\mathrm{e}^{-\frac{2g^2}{\omega^2}}L_n\left(\frac{4g^2}{\omega^2}\right) \\ (\mathrm{K}) & \frac{\Delta}{2}\sigma\frac{n!}{(2K)_n}(1+|\kappa|^2)^{-K-n}F_n(|\kappa|^2:2K) \qquad \text{where} \quad \kappa = \sinh(x) \\ (\mathrm{J}) & \frac{\Delta}{2}\sigma\frac{n!}{{}_{2J}P_n}(1-|\kappa|^2)^{J-n}F_n(|\kappa|^2:2J) \qquad \text{where} \quad \kappa = \sin(x) \end{cases} \tag{56}$$

from (41) and the results in section 3.1. Now let us solve (48)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_0 = \frac{\Delta}{2}H_F\Psi_0 = \frac{\Delta}{2}(H_F' + H_F'')\Psi_0.$$

For that if we set $\Psi_0(t)$ as

$$\Psi_0(t) = \sum_\sigma\sum_n \mathrm{e}^{-\mathrm{i}tE_{n,\sigma}}a_{n,\sigma}(t)|\{\sigma,\psi_n\}\rangle \tag{57}$$

then we have a set of complicated equations with respect to $\{a_{n,\sigma}\}$, see [12]. But it is almost impossible to solve them. Therefore, we make a daring assumption: for $m < n$

$$\Psi_0(t) = \sum_\sigma \mathrm{e}^{-\mathrm{i}tE_{m,\sigma}}a_{m,\sigma}(t)|\{\sigma,\psi_m\}\rangle + \sum_\sigma \mathrm{e}^{-\mathrm{i}tE_{n,\sigma}}a_{n,\sigma}(t)|\{\sigma,\psi_n\}\rangle. \tag{58}$$

That is, we consider only two terms with respect to $\{n|n \geqslant 0\}$. After some algebras we obtain

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}a_{m,\sigma} = \frac{\Delta}{2}\sum_{\sigma'} \mathrm{e}^{-\mathrm{i}t(E_{n,\sigma'}-E_{m,\sigma})}\,\mathrm{e}^{\mathrm{i}t\Omega(m-n)}\left\{\langle\langle m|\,\mathrm{e}^{x(L_+-L_-)}|n\rangle\rangle\frac{\sigma}{2} + \langle\langle m|\,\mathrm{e}^{-x(L_+-L_-)}|n\rangle\rangle\frac{\sigma'}{2}\right\}a_{n,\sigma'}$$

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}a_{n,\sigma} = \frac{\Delta}{2}\sum_{\sigma'} \mathrm{e}^{-\mathrm{i}t(E_{m,\sigma'}-E_{n,\sigma})}\,\mathrm{e}^{\mathrm{i}t\Omega(n-m)}$$

$$\times\left\{\langle\langle n|\,\mathrm{e}^{x(L_+-L_-)}|m\rangle\rangle\frac{\sigma}{2} + \langle\langle n|\,\mathrm{e}^{-x(L_+-L_-)}|m\rangle\rangle\frac{\sigma'}{2}\right\}a_{m,\sigma'}. \tag{59}$$

But we cannot still solve the above equations exactly (see the appendix), so let us make the so-called rotating-wave approximation. The resonance condition is

$$-(E_{n,\sigma'} - E_{m,\sigma}) + (m-n)\Omega = 0 \quad\Longrightarrow\quad E_{n,\sigma'} - E_{m,\sigma} = (m-n)\Omega \tag{60}$$

for some σ and σ', and we reject the remaining terms in (59). Then we obtain simple equations:

Interband transition case ($\sigma \neq \sigma'$). $E_{n,-\sigma} - E_{m,\sigma} = (m-n)\Omega$

$$\begin{aligned}
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}a_{m,\sigma} &= \frac{\Delta}{2}\left\{\langle\langle m|\,\mathrm{e}^{x(L_+-L_-)}|n\rangle\rangle\frac{\sigma}{2} - \langle\langle m|\,\mathrm{e}^{-x(L_+-L_-)}|n\rangle\rangle\frac{\sigma}{2}\right\}a_{n,-\sigma}\\
&= \frac{\Delta}{2}\sigma\langle\langle m|\sinh(x(L_+-L_-))|n\rangle\rangle a_{n,-\sigma}\\
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}a_{n,-\sigma} &= \frac{\Delta}{2}\sum_{\sigma'}\left\{-\langle\langle n|\,\mathrm{e}^{x(L_+-L_-)}|m\rangle\rangle\frac{\sigma}{2} + \langle\langle n|\,\mathrm{e}^{-x(a^\dagger-a)}|m\rangle\rangle\frac{\sigma}{2}\right\}a_{m,\sigma}\\
&= -\frac{\Delta}{2}\sigma\langle\langle n|\sinh(x(L_+-L_-))|m\rangle\rangle a_{m,\sigma}.
\end{aligned} \tag{61}$$

Intraband transition case ($\sigma = \sigma'$). $E_{n,\sigma} - E_{m,\sigma} = (m-n)\Omega$

$$\begin{aligned}
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}a_{m,\sigma} &= \frac{\Delta}{2}\left\{\langle\langle m|\,\mathrm{e}^{x(L_+-L_-)}|n\rangle\rangle\frac{\sigma}{2} + \langle\langle m|\,\mathrm{e}^{-x(L_+-L_-)}|n\rangle\rangle\frac{\sigma}{2}\right\}a_{n,\sigma}\\
&= \frac{\Delta}{2}\sigma\langle\langle m|\cosh(x(L_+-L_-))|n\rangle\rangle a_{n,\sigma}\\
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}a_{n,\sigma} &= \frac{\Delta}{2}\sum_{\sigma'}\left\{\langle\langle n|\,\mathrm{e}^{x(L_+-L_-)}|m\rangle\rangle\frac{\sigma}{2} + \langle\langle n|\,\mathrm{e}^{-x(L_+-L_-)}|m\rangle\rangle\frac{\sigma}{2}\right\}a_{m,\sigma}\\
&= \frac{\Delta}{2}\sigma\langle\langle n|\cosh(x(L_+-L_-))|m\rangle\rangle a_{m,\sigma}.
\end{aligned} \tag{62}$$

For simplicity we set

$$\mathcal{R} = \Delta\langle\langle n|\sinh(x(L_+-L_-))|m\rangle\rangle \qquad \mathcal{R}' = \Delta\langle\langle n|\cosh(x(L_+-L_-))|m\rangle\rangle \tag{63}$$

then

$$\Delta\langle\langle m|\sinh(x(L_+-L_-))|n\rangle\rangle = -\mathcal{R} \qquad \Delta\langle\langle m|\cosh(x(L_+-L_-))|n\rangle\rangle = \mathcal{R}'.$$

These are two Rabi frequencies as shown in the following. It is important that Rabi frequencies in our models are given by *matrix elements of coherent and generalized coherent operators*!

By making use of the results in section 3 and (38), (39), (40) we have

$$(\mathrm{N})\quad \begin{cases} \mathcal{R} = \frac{\Delta}{2}\sqrt{\frac{m!}{n!}}\left(\frac{2g}{\omega}\right)^{n-m}\mathrm{e}^{-\frac{2g^2}{\omega^2}}L_m{}^{(n-m)}\left(\frac{4g^2}{\omega^2}\right)\{1-(-1)^{n-m}\}\\ \mathcal{R}' = \frac{\Delta}{2}\sqrt{\frac{m!}{n!}}\left(\frac{2g}{\omega}\right)^{n-m}\mathrm{e}^{-\frac{2g^2}{\omega^2}}L_m{}^{(n-m)}\left(\frac{4g^2}{\omega^2}\right)\{1+(-1)^{n-m}\}\end{cases} \tag{64}$$

$$
\text{(K)}\quad \begin{cases} \mathcal{R} = \frac{\Delta}{2}\sqrt{\frac{n!m!}{(2K)_n(2K)_m}}\kappa^{n-m}(1+|\kappa|^2)^{-K-\frac{n+m}{2}} {F_m}^{(n-m)}(|\kappa|^2 : 2K)\{1-(-1)^{n-m}\} \\ \mathcal{R}' = \frac{\Delta}{2}\sqrt{\frac{n!m!}{(2K)_n(2K)_m}}\kappa^{n-m}(1+|\kappa|^2)^{-K-\frac{n+m}{2}} {F_m}^{(n-m)}(|\kappa|^2 : 2K)\{1+(-1)^{n-m}\} \end{cases}
$$

$$
\text{where}\quad \kappa = \sinh(x) \qquad \text{with}\quad x = \tanh^{-1}\left(\frac{2g}{\omega}\right) \tag{65}
$$

$$
\text{(J)}\quad \begin{cases} \mathcal{R} = \frac{\Delta}{2}\sqrt{\frac{n!m!}{{}_{2J}P_{n\,2J}P_m}}\kappa^{n-m}(1-|\kappa|^2)^{J-\frac{n+m}{2}} {F_m}^{(n-m)}(|\kappa|^2 : 2J)\{1-(-1)^{n-m}\} \\ \mathcal{R}' = \frac{\Delta}{2}\sqrt{\frac{n!m!}{{}_{2J}P_{n\,2J}P_m}}\kappa^{n-m}(1-|\kappa|^2)^{J-\frac{n+m}{2}} {F_m}^{(n-m)}(|\kappa|^2 : 2J)\{1+(-1)^{n-m}\} \end{cases}
$$

$$
\text{where}\quad \kappa = \sin(x) \qquad \text{with}\quad x = \tan^{-1}\left(\frac{2g}{\omega}\right) \tag{66}
$$

From these we find a constraint between m and n:

Interband case. $n - m = 2N - 1 \Longrightarrow n = m + 2N - 1$ for $N \in \mathbf{N}$,
Intraband case. $n - m = 2N \Longrightarrow n = m + 2N$ for $N \in \mathbf{N}$.

Now let us solve (61) and (62).

$$
\begin{aligned}
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a_{m,\sigma} \\ a_{n,-\sigma} \end{pmatrix} &= \begin{pmatrix} 0 & -\sigma\frac{\mathcal{R}}{2} \\ -\sigma\frac{\mathcal{R}}{2} & 0 \end{pmatrix}\begin{pmatrix} a_{m,\sigma} \\ a_{n,-\sigma} \end{pmatrix} \\
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a_{m,\sigma} \\ a_{n,\sigma} \end{pmatrix} &= \begin{pmatrix} 0 & \sigma\frac{\mathcal{R}'}{2} \\ \sigma\frac{\mathcal{R}'}{2} & 0 \end{pmatrix}\begin{pmatrix} a_{m,\sigma} \\ a_{n,\sigma} \end{pmatrix}
\end{aligned} \tag{67}
$$

so their solutions are given by

$$
\begin{aligned}
\begin{pmatrix} a_{m,\sigma}(t) \\ a_{n,-\sigma}(t) \end{pmatrix} &= \begin{pmatrix} \cos\left(\frac{\mathcal{R}}{2}t\right) & \mathrm{i}\sigma\sin\left(\frac{\mathcal{R}}{2}t\right) \\ \mathrm{i}\sigma\sin\left(\frac{\mathcal{R}}{2}t\right) & \cos\left(\frac{\mathcal{R}}{2}t\right) \end{pmatrix}\begin{pmatrix} a_{m,\sigma}(0) \\ a_{n,-\sigma}(0) \end{pmatrix} \\
\begin{pmatrix} a_{m,\sigma}(t) \\ a_{n,\sigma}(t) \end{pmatrix} &= \begin{pmatrix} \cos\left(\frac{\mathcal{R}'}{2}t\right) & -\mathrm{i}\sigma\sin\left(\frac{\mathcal{R}'}{2}t\right) \\ -\mathrm{i}\sigma\sin\left(\frac{\mathcal{R}'}{2}t\right) & \cos\left(\frac{\mathcal{R}'}{2}t\right) \end{pmatrix}\begin{pmatrix} a_{m,\sigma}(0) \\ a_{n,\sigma}(0) \end{pmatrix}.
\end{aligned} \tag{68}
$$

We have obtained some solutions under the rotating-wave approximation. Now it may be suited to compare our results with a recent experimental finding in [11], but this is beyond our scope (see [12]).

Let us conclude this section by a comment. Our ansatz (58) to solve the equation is too restrictive. We want to use (57) to solve the equation, but it is very hard at this stage.

Problem. *Find more dynamic methods*!

5. Quantum computation

Let us reconsider the results in the preceding section in the light of quantum computation. Recall once more that the following arguments are based on the *rotating-wave approximation*.

Interband case ($\sigma = \mathbf{1}$)

$$
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a_{m,1} \\ a_{m,-1} \\ a_{n,1} \\ a_{n,-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\frac{\mathcal{R}}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\mathcal{R}}{2} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_{m,1} \\ a_{m,-1} \\ a_{n,1} \\ a_{n,-1} \end{pmatrix}. \tag{69}
$$

The solution is

$$\begin{pmatrix} a_{m,1}(t) \\ a_{m,-1}(t) \\ a_{n,1}(t) \\ a_{n,-1}(t) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\mathcal{R}}{2}t\right) & 0 & 0 & \mathrm{i}\sin\left(\frac{\mathcal{R}}{2}t\right) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{i}\sin\left(\frac{\mathcal{R}}{2}t\right) & 0 & 0 & \cos\left(\frac{\mathcal{R}}{2}t\right) \end{pmatrix} \begin{pmatrix} a_{m,1}(0) \\ a_{m,-1}(0) \\ a_{n,1}(0) \\ a_{n,-1}(0) \end{pmatrix}. \tag{70}$$

Interband case ($\sigma = -\mathbf{1}$)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a_{m,1} \\ a_{m,-1} \\ a_{n,1} \\ a_{n,-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathcal{R}}{2} & 0 \\ 0 & \frac{\mathcal{R}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{m,1} \\ a_{m,-1} \\ a_{n,1} \\ a_{n,-1} \end{pmatrix}. \tag{71}$$

The solution is

$$\begin{pmatrix} a_{m,1}(t) \\ a_{m,-1}(t) \\ a_{n,1}(t) \\ a_{n,-1}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{\mathcal{R}}{2}t\right) & -\mathrm{i}\sin\left(\frac{\mathcal{R}}{2}t\right) & 0 \\ 0 & -\mathrm{i}\sin\left(\frac{\mathcal{R}}{2}t\right) & \cos\left(\frac{\mathcal{R}}{2}t\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{m,1}(0) \\ a_{m,-1}(0) \\ a_{n,1}(0) \\ a_{n,-1}(0) \end{pmatrix}. \tag{72}$$

Intraband case ($\sigma = \mathbf{1}$)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a_{m,1} \\ a_{m,-1} \\ a_{n,1} \\ a_{n,-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\mathcal{R}'}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\mathcal{R}'}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{m,1} \\ a_{m,-1} \\ a_{n,1} \\ a_{n,-1} \end{pmatrix}. \tag{73}$$

The solution is

$$\begin{pmatrix} a_{m,1}(t) \\ a_{m,-1}(t) \\ a_{n,1}(t) \\ a_{n,-1}(t) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\mathcal{R}'}{2}t\right) & 0 & -\mathrm{i}\sin\left(\frac{\mathcal{R}'}{2}t\right) & 0 \\ 0 & 1 & 0 & 0 \\ -\mathrm{i}\sin\left(\frac{\mathcal{R}'}{2}t\right) & 0 & \cos\left(\frac{\mathcal{R}'}{2}t\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{m,1}(0) \\ a_{m,-1}(0) \\ a_{n,1}(0) \\ a_{n,-1}(0) \end{pmatrix}. \tag{74}$$

Intraband case ($\sigma = -\mathbf{1}$)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a_{m,1} \\ a_{m,-1} \\ a_{n,1} \\ a_{n,-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\mathcal{R}'}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathcal{R}'}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{m,1} \\ a_{m,-1} \\ a_{n,1} \\ a_{n,-1} \end{pmatrix}. \tag{75}$$

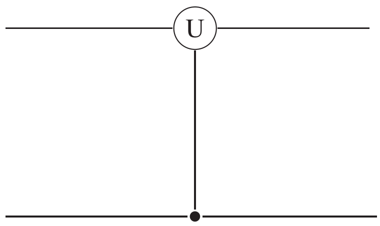
The solution is

$$\begin{pmatrix} a_{m,1}(t) \\ a_{m,-1}(t) \\ a_{n,1}(t) \\ a_{n,-1}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{\mathcal{R}'}{2}t\right) & 0 & \mathrm{i}\sin\left(\frac{\mathcal{R}'}{2}t\right) \\ 0 & 0 & 1 & 0 \\ 0 & \mathrm{i}\sin\left(\frac{\mathcal{R}'}{2}t\right) & 0 & \cos\left(\frac{\mathcal{R}'}{2}t\right) \end{pmatrix} \begin{pmatrix} a_{m,1}(0) \\ a_{m,-1}(0) \\ a_{n,1}(0) \\ a_{n,-1}(0) \end{pmatrix}. \tag{76}$$

If we can identify (58) with an element in two-qubit space

$$a_{m,1}(t)|00\rangle + a_{m,-1}(t)|01\rangle + a_{n,1}(t)|10\rangle + a_{n,-1}(t)|11\rangle \in \mathbf{C}^2 \otimes \mathbf{C}^2 \tag{77}$$

where $\mathbf{C}^2 = \mathrm{Vect}_{\mathbf{C}}\{|0\rangle, |1\rangle\}$, then the solutions (70), (72), (74), (76) are kinds of controlled unitary operations (gates) which play a crucial role in quantum computation, see for example [16]. For example, (76) is just one of the controlled unitary gates expressed graphically as



We note here that controlled unitary gates above are written down as

$$\text{C-Unitary} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & 0 & u_{12} \\ 0 & 0 & 1 & 0 \\ 0 & u_{21} & 0 & u_{22} \end{pmatrix} \tag{78}$$

for

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in U(2).$$

A comment is in order. The same subject is considered in [17] (and [18]). However, the authors in them treated it in the weak coupling regime, while we treated it in the strong coupling regime. It is very interesting to investigate a deep relation (connection) between them. See also [19] as an interesting paper and its references.

We would like to conclude this section by stating that a possible realization of our model could be found in Josephson junctions [11] and ion cavities [17–19]. Further study will be required.

6. Discussion

One of the motivations of this study is to apply our results to holonomic quantum computation developed by the Italian group (Pachos, Rasetti and Zanardi) and the author, see [20–28] and most recently [29, 30].

In this theory we usually use the effective Hamiltonian of a single-mode field of Kerr medium

$$H_0 = XN(N-1) \qquad N = a^\dagger a \quad \text{where} \quad X \quad \text{is a constant} \tag{79}$$

as a background and the real Hamiltonian is in the one-qubit case given by

$$H(z, w) = W(z, w)H_0W^{-1}(z, w) \tag{80}$$

where W is a product of coherent operator $U(z)$ and squeezed one $S(w)$ in section 2. In the above Hamiltonian H_0 the zero-eigenvalue is two-fold degenerate with eigenvectors $|0\rangle$ and $|1\rangle$. We set $|vac\rangle = (|0\rangle, |1\rangle)$. Then we can construct a connection form $\mathcal{A}$ on the parameter space $\{(z, w) \in \mathbf{C}^2\}$ as

$$\mathcal{A} = \langle vac|W^{-1}\,\mathrm{d}W|vac\rangle \tag{81}$$

from (80) where $\mathrm{d} = \mathrm{d}z\frac{\partial}{\partial z} + \mathrm{d}w\frac{\partial}{\partial w}$. By making use of this connection we can construct a holonomy group $\mathrm{Hol}(\mathcal{A})$ ($\subseteq U(2)$) which is in this case equal to $U(2)$. In holonomic quantum computation we use this holonomy group as unitary operations in quantum computation. The point at issue is that we do not use the full property of the Hamiltonian but only the property of the zero-eigenvalue.

The Hamiltonian H_F in (49)

$$H_F = \frac{\Delta}{2}{U_0}^{-1}(\sigma_3 \otimes \mathbf{1}_L)U_0 \qquad \text{where} \quad L = (\mathrm{N}) \quad \text{or} \quad (\mathrm{K}) \quad \text{or} \quad (\mathrm{J})$$

is very similar to (80). This system is always two-fold degenerate. Then a natural question arises:

Problem. *Is it possible to perform a holonomic quantum computation by combining the systems* $\{(\mathrm{N}), (\mathrm{K}), (\mathrm{J})\}$?

This is a very interesting and challenging problem.

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Appendix. On equations (59)

Here let us write down full equations of (59) with matrix equation form:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{a}_m \\ \mathbf{a}_n\end{pmatrix} = \begin{pmatrix}\mathbf{0} & A \\ A^\dagger & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{a}_m \\ \mathbf{a}_n\end{pmatrix} \tag{A.1}$$

where

$$\mathbf{a}_k = \begin{pmatrix} a_{k,1} \\ a_{k,-1}\end{pmatrix} \qquad \text{for} \quad k = m, n$$

and

$$\begin{aligned} A \equiv A(t) &= \begin{pmatrix} \frac{\mathcal{R}'}{2}\,\mathrm{e}^{\mathrm{i}t(-E_{n,1}+E_{m,1}+\Omega(m-n))} & -\frac{\mathcal{R}}{2}\,\mathrm{e}^{\mathrm{i}t(-E_{n,-1}+E_{m,1}+\Omega(m-n))} \\ \frac{\mathcal{R}}{2}\,\mathrm{e}^{\mathrm{i}t(-E_{n,1}+E_{m,-1}+\Omega(m-n))} & -\frac{\mathcal{R}'}{2}\,\mathrm{e}^{\mathrm{i}t(-E_{n,-1}+E_{m,-1}+\Omega(m-n))}\end{pmatrix} \\ &= \mathrm{e}^{\mathrm{i}t\Omega(m-n)}\begin{pmatrix} \frac{\mathcal{R}'}{2}\,\mathrm{e}^{\mathrm{i}t(-E_{n,1}+E_{m,1})} & -\frac{\mathcal{R}}{2}\,\mathrm{e}^{\mathrm{i}t(E_{n,1}+E_{m,1})} \\ \frac{\mathcal{R}}{2}\,\mathrm{e}^{\mathrm{i}t(-E_{n,1}-E_{m,1})} & -\frac{\mathcal{R}'}{2}\,\mathrm{e}^{\mathrm{i}t(E_{n,1}-E_{m,1})}\end{pmatrix} \end{aligned} \tag{A.2}$$

because $E_{k,-\sigma} = -E_{k,\sigma}$.

We can give (A.1) a formal solution by infinite series (called Dyson series in theoretical physics). Then we meet secular terms.

For example, let us consider the following simple equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}a = \mathrm{e}^{\mathrm{i}\omega t}a \qquad \text{with} \quad a(0) = c.$$

The solution is given by

$$a(t) = \begin{cases} c\exp\left(\frac{\mathrm{e}^{\mathrm{i}\omega t}-1}{\mathrm{i}\omega}\right) & \omega \neq 0 \\ c\,\mathrm{e}^t & \omega = 0 \end{cases}$$

That is, we meet the secular term.

Therefore, we have learnt how to handle (simple) secular terms called the renormalization group method (approach), see [31] for a general introduction.

Frasca in [32] has applied this method to the above equation. The conclusion is interesting, but seems to be rather involved. We are now reconsidering his approach. Therefore let us present.

Problem. *Solve this matrix equation completely*!

References

[1] Klauder J R and Skagerstam Bo-S (ed) 1985 *Coherent States* (Singapore: World Scientific)

[2] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)

[3] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)

[4] Funahashi K, Kashiwa T, Sakoda S and Fujii K 1995 Coherent states, path integral, and semiclassical approximation *J. Math. Phys.* **36** 3232

[5] Funahashi K, Kashiwa T, Sakoda S and Fujii K 1995 Exactness in the Wentzel–Kramers–Brillouin approximation for some homogeneous spaces *J. Math. Phys.* **36** 4590

[6] Fujii K, Kashiwa T and Sakoda S 1996 Coherent states over Grassmann manifolds and the WKB exactness in path integral *J. Math. Phys.* **37** 567

[7] Fujii K 2001 Basic properties of coherent and generalized coherent operators revisited *Mod. Phys. Lett.* A **16** 1277 (*Preprint* quant-ph/0009012)

[8] Fujii K 2000 Note on extended coherent operators and some basic properties *Preprint* quant-ph/0009116

[9] Fujii K 2001 Introduction to coherent states and quantum information theory *Preprint* quant-ph/0112090

[10] Meystre P and Sargent III M 1990–1991 *Elements of Quantum Optics* (Berlin: Springer)

[11] Nakamura Y, Pashkin Yu A and Tsai J S 2001 Rabi oscillations in a Josephson-junction charge two-level system *Phys. Rev. Lett.* **87** 246601

[12] Frasca M 2002 Rabi oscillations and macroscopic quantum superposition states *Phys. Rev.* A **66** 023810 (*Preprint* quant-ph/0111134)

[13] Frasca M 2001 Dephasing by two-level systems at zero temperature by unitary evolution *Preprint* cond-mat/0112253 (*Physica* E at press)

[14] Fujii K 2002 Matrix elements of generalized coherent operators *Preprint* quant-ph/0202081

[15] Schleich W P 2001 *Quantum Optics in Phase Space* (New York: Wiley–VCH)

[16] Fujii K 2002 Introduction to Grassmann manifolds and quantum computation *J. Appl. Math.* **2** 371–405 (*Preprint* quant-ph/0103011)

[17] Montoe C, Leibfried D, King B E, Meekhof D M, Itano W M and Wineland D J 1996 Simplified quantum logic with trapped ions *Preprint* quant-ph/9612042

[18] Itano W M, Montoe C, Meekhof D M, Leibfried D, King B E and Wineland D J 1997 Quantum harmonic oscillator state synthesis and analysis *Preprint* quant-ph/9702038

[19] Moya-Cessa H, Vidiella A, Roversi J A, Freitas D S and Dutra S M 1999 Long-time-scale revivals in ion traps *Phys. Rev.* A **59** 2518

[20] Zanardi P and Rasetti M 1999 Holonomic quantum computation *Phys. Lett.* A **264** 94 (*Preprint* quant-ph/9904011)

[21] Pachos J, Zanardi P and Rasetti M 2000 Non-Abelian Berry connections for quantum computation *Phys. Rev.* A **61** 010305(R) (*Preprint* quant-ph/9907103)

[22] Pachos J and Chountasis S 2000 Optical holonomic quantum computer *Phys. Rev.* A **62** 052318 (*Preprint* quant-ph/9912093)

[23] Pachos J and Zanardi P 2000 Quantum holonomies for quantum computing *Preprint* quant-ph/0007110

[24] Fujii K 2000 Note on coherent states and adiabatic connections, curvatures *J. Math. Phys.* **41** 4406 (*Preprint* quant-ph/9910069)

[25] Fujii K 2001 Mathematical foundations of holonomic quantum computer *Rep. Math. Phys.* **48** 75 (*Preprint* quant-ph/0004102)

[26] Fujii K 2000 More on optical holonomic quantum computer *Preprint* quant-ph/0005129

[27] Fujii K 2000 Mathematical foundations of holonomic quantum computer II *Preprint* quant-ph/00101102

[28] Fujii K 2001 From geometry to quantum computation *Proc. 2nd Int. Symp. Quantum Theory and Symmetries* (Singapore: World Scientific) (*Preprint* quant-ph/0107128) at press

[29] Lucarelli D 2001 Chow's theorem and universal holonomic quantum computation *Preprint* quant-ph/0111078

[30] Lucarelli D 2002 Control algebra for holonomic quantum computation with squeezed coherent states *Preprint* quant-ph/0202055

[31] Ei S, Fujii K and Kunihiro T 2000 Renormalization-group method for reduction of evolution equations; invariant manifolds and envelopes *Ann. Phys., NY* **280** 236 (*Preprint* hep-th/9905088)

[32] Frasca M 1998 Theory of quantum resonance: a renormalization-group approach *Phys. Rev.* A **58** 771